

# Rigorous Comparison of Gravimetry Employing Atom Interferometers and the Measurement of Gravitational Time Dilation

C. S. Unnikrishnan<sup>1</sup> and G. T. Gillies<sup>2</sup>

<sup>1</sup>*Gravitation Group, Tata Institute of Fundamental Research,  
Homi Bhabha Road, Mumbai - 400 005, India*

<sup>2</sup>*School of Engineering and Applied Science,  
University of Virginia, Charlottesville, VA 22904-4746, USA*  
E-mail address: <sup>1</sup>unni@tifr.res.in, <sup>2</sup>gtg@virginia.edu

## Abstract

We present a gravitationally rigorous and clear answer, in the negative, to the question whether gravimetry with atom interferometers is equivalent to the measurement of the relative gravitational time dilation of two clocks separated in space. Though matter and light waves, quantum states and oscillator clocks are quantum synonymous through the Planck-Einstein-de Broglie relations and the equivalence principle, there are crucial differences in the context of gravitation theories.

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The relative time dilation of two clocks at different locations in a gravitational field was predicted and calculated by Einstein in 1911 using the principle of equivalence and the Doppler shift in motion, well before the full theory of general theory of relativity was formulated [1]. Einstein's insightful interpretation of the shift in the frequency of the radiation as a consequence of the clocks running at different rates is the connection between the phenomenon of gravitational redshift and the gravitation time dilation. Since frequency is a number referred to a clock, only a change in the rate of the clock can change the frequency. Measurement of gravitational redshift compares the frequency or energy of the radiation between two points directly as in the Pound-Rebka experiment, whereas gravitational time dilation is best measured by comparing the accumulated time difference between two physical clocks in a gravitational field with radiation or an electromagnetic contact serving only as a means of communication, needed only at the beginning and the end of the comparison, in those cases when the clocks cannot be transported for comparison.

An important unifying idea is that *once the frequency of an oscillator is specified, its progressive phase is equivalent to time*; there is no difference between physical time and physical phase if an oscillator is used as the basis of time measurement. This implies the universality of the gravitational effect on all time dependent phenomena [2]. Gravity couples to the total energy of a physical system, and energy is proportional to a 'frequency' in quantum physics and optics and hence the same physical system in two different gravitational potentials have different time evolutions and accumulated phase. Since the phase of an oscillator of every kind is equivalent to time, states, photon, and matter waves can all be interpreted as 'clocks' in quantum mechanics, apart from real physical clocks themselves [2]. A stationary state of definite energy in quantum mechanics is an 'oscillator', with free time evolution factor  $\exp(-iEt/\hbar)$ , with a 'frequency' given by the relation  $\nu = E/h$ . Light obeys the relation  $E = h\nu$  in spite of being treated as discrete photons. And a clock, like an atomic clock, is based on transitions that obey  $\Delta E = h\nu$ . However, there is a price to pay for this universality – the associated wave is an abstract and unobservable entity, manifesting only through its relation to relevant probabilities, except perhaps in the case of low frequency radiation where the interaction of the 'wave' on charged particles can be seen directly. We stress this point because it is important in the rigorous and correct interpretation of what the measurement of time in a gravitational field means. It is only in a space-time interpretation of quantum physics, as in the de

Broglie-Bohm theory for example, spatial ontological status can be ascribed to the quantum wave.

The analysis of quantum dynamics in weak (laboratory) gravitational fields reduces to the relations  $E_g = -E\phi_g/c^2$  and the phase  $d\varphi = E_g(x, t)dt/\hbar$  accumulated over time duration  $dt$ , where  $E$  is total energy of the physical system and the gravitational potential  $\phi_g$  is simply related to the metric component  $g_{00}$ . The gravitational part of the quantum evolution is determined by the integrated phase over the path given by  $\Delta_g(x) = \int E_g(x)dt/\hbar$ .

The gravitational energy of a massive particle is  $E_g = -m_g\phi_g$  and therefore, the quantum phase depends explicitly on the gravitational mass. However, as stressed in references [3, 4], there is full compatibility with the classical equivalence principle. Since the quantum phase is proportional to the product of this energy and the time spent in the potential,  $t \simeq l/v$ , where  $l$  is the spatial scale and  $v$  the velocity of the particle, the accumulated phase in each path is

$$\Delta_g \simeq E_g t / \hbar = -m_g \phi_g l / v \hbar \quad (1)$$

We can rewrite this expression, using the relation between the inertial mass and the de Broglie wavelength in quantum theory,  $\lambda_{dB} = 2\pi\hbar/m_i v$ , as

$$\Delta_g = -m_g \phi_g l m_i \lambda_{dB} / \hbar^2 \quad (2)$$

or as

$$\Delta_g = - \left( \frac{m_g}{m_i} \right) \left( \frac{\phi_g}{v^2} \right) \left( \frac{l}{\lambda_{dB}} \right) = - \left( \frac{E_g}{2E_{kin}} \right) \left( \frac{l}{\lambda_{dB}} \right) \quad (3)$$

The expression is particularly interesting due to the scaling expressed in terms of the kinetic energy and the wavelength, and more importantly due to the appearance of the ratio of the gravitational and the inertial mass.

For atomic clocks that work with a transition frequency  $\nu$  between two stationary states,  $E = h\nu$ , the accumulated gravitational time dilation is  $\delta T = \Delta_g / 2\pi\nu$  and the relative time dilation for *two clocks* at points  $x_1$  and  $x_2$  is given by

$$\delta T_r = \frac{\Delta_g(x_1) - \Delta_g(x_2)}{2\pi\nu} = T [\phi_g(x_1) - \phi_g(x_2)] / c^2 = T g l / c^2 \quad (4)$$

Here,  $l = x_1 - x_2$  and  $T$  is the total duration of comparison. We have assumed that the potential over each of the relevant path is constant, for simplicity. This is the standard general relativistic expression.

Several points about equation 4 is worth mentioning. The accumulated time dilation is independent of the properties of the clock, especially its mass etc. This is a consequence of implicitly assuming the equivalence principle exactly, as we will show. It is also independent of the frequency of the oscillator, for the same reason. The time dilation is the difference between the phases accumulated over each path (or position in the gravitational field), usually by two different clocks, each of which can serve as an independent physical clock. If phase difference is the measured quantity, the smallest time dilation that can be resolved, or the precision of the measurement, is inversely proportional to the frequency of the ‘oscillator’, a point that is important for the discussion here.

We now show that the equation 4 requires a subtle modification when the equivalence principle is not assumed in its exact form. This is relevant for experiments that test small violations of the equivalence principle. The gravitational coupling is the gravitational mass whereas the Shrodinger evolution equation and relations of the form  $E = mc^2 = h\nu$  refers to the inertial mass. Keeping this distinction in mind and referring to massive particles, the gravitationally modified phase is

$$\Delta_g(x) = \int E_g(x)dt/\hbar = - \int m_g\phi_g dt/\hbar \quad (5)$$

Then the expression for time dilation should be written as

$$\delta T_r = \frac{\Delta_g(x_1) - \Delta_g(x_2)}{2\pi\nu} = \left(\frac{m_g}{m_i}\right) Tgl/c^2 \quad (6)$$

The presence of the term  $\Delta_g(x_1) - \Delta_g(x_2)$  referring to the accumulated phase difference between two oscillator clocks is tempting enough to interpret the differential phase accumulated in situations of two-path interferometry as a differential clock delay. For example, the case of a single photon in a double slit experiment in which the slits are positioned vertically in a gravitational field involves gravitationally induced phase difference between the two possible paths to the detector point on screen,

$$\Delta_g = -\frac{1}{\hbar} \int dt (h\nu/c^2) (\phi_g(x_1) - \phi_g(x_2)) \quad (7)$$

This is negligibly small in laboratory situations. In principle, it can be made measurable as a shift of the fringe position proportional to the gravitational

field. In contrast, if massive atoms or neutrons are used in double-slit configuration in a gravitational field, the ‘mass term’ is  $10^9$  to  $10^{11}$  times larger, being the ratio  $mc^2/h\nu$ . This is the enormous advantage of matter wave interferometry in metrology, even though some of this advantage gets lost in the fact that the extent of the path is usually much smaller than what is possible in the case of optical interferometry. An important question, raised by a recent debate [6, 7], is whether the measurement of the gravitationally induced phase in such an experiment can be interpreted as the measurement of gravitational time dilation between two clocks at different points in a gravitational field. Müller *et al* claimed that the quantum phase difference accrued between the two quantum states of the same particle in situations of atom interferometry, as in gravimetry with an atom interferometer, is equivalent to the measurement of the gravitational time dilation between two clocks [6]. In their view, the quantum phase over each path over a duration  $T$  can be re-written in terms of the Compton frequency  $\omega_c = mc^2/\hbar$  as

$$\Delta_g(x) = -m\phi_g(x)T/\hbar = -\frac{mc^2}{\hbar c^2}\phi_g(x)T = -\omega_c\phi_g(x)T/c^2 \quad (8)$$

The smallest time dilation factor than can be measured is then  $\delta T = \Delta_g/\omega_c$ . Therefore, if it is legitimate to imagine the moving atom as a real ‘Compton wave clock’ in space and time, the expression for the gravitational phase shift written in terms of the Compton frequency suggests exceptional and unprecedented sensitivity for the measurement of gravitational time dilation, since the Compton frequency is more than  $10^{10}$  times larger than even optical frequencies of modern atomic clocks (this gain is the same as the ratio of the rest mass to photon energy, noted earlier). Based on this assumption and the results of gravimetry employing atom interferometry performed about a decade ago [8, 9], Müller *et al* claimed [6] enormous improvement of the measurement of gravitational time dilation, by a factor of about  $10^6$ .

The comment of disagreement by Wolf *et al* focussed on dismissing the notion of treating the moving atom as a Compton wave in space [7]. They pointed out that the expression for the measured differential phase shift does not contain the Compton frequency, being independent of the mass of the atom. The debate saw each side holding on to their views, supported on both sides by similar arguments by other authors [10, 11, 12]. We aim and achieve a gravitationally rigorous and physically justified resolution of the issue. We do this by proving the following main results: a) the correct expression for phase difference in an atom interferometer involves the ratio of gravitational and

inertial masses, and also the de Broglie wavelength  $\lambda_{dB}$ , and not the Compton wavelength, b) The phase shift is essentially the ratio of free fall distance in the gravitational field  $g$  and the de Broglie wavelength, and c) the mass term does not drop out, since the de Broglie wavelength  $\lambda_{dB} \sim 1/m$  explicitly appears. However this can be replaced with the differential momentum  $\kappa$  of a two-photon Raman transition used in the experiment, since  $\lambda_{dB} = 2\pi/\kappa$ , to give the impression that it is independent of the mass. We also point out two other relevant points in support; 1) a quantum system in a superposition of two eigen states that develop a spatial separation in state space is not equivalent to physical clocks that are spatially separated, each of which can be accessed for comparison with a primary clock, 2) the fact that it is the de Broglie wavelength that is associated with the atoms in propagation can be directly checked in a conventional double slit experiment, revealing the spatial fringes scaled to the de Broglie wavelength. The associated frequency obeys the relation  $E = h\nu$  with  $E = p^2/2m$  and not with  $E = mc^2$ .

Quantum dynamics of a system that can be in a superposition of two states can be used for atom interferometry by entangling the states with positional degree of freedom. In the case of atom interferometry with a two-state atom, creating superposition of the two states involves laser excitation that attaches different momenta to the two states, which results in evolution that associates two different spatial positions with the two states. For example, coherent resonant excitation by a laser with sufficient pulse duration ( $\pi/2$  in terms of the inverse Rabi frequency) that creates an equal superposition  $(|g\rangle + |e\rangle)/\sqrt{2}$  of ground and excited states  $|g\rangle$  and  $|e\rangle$  starting with the ground state, generates the detailed entangled state  $|g, 0\rangle + |e, \hbar k\rangle/\sqrt{2}$ . The difference in momentum  $\hbar k$  develops into a separation of the quantum states in spatial coordinates, visible on measurement as two distributions of atoms centred on these spatial positions, separated by  $\Delta x = \hbar k t/m$ . The differential quantum phase of such entangled states in a gravitational field is the basis of gravimeters and inertial sensors employing atom interferometry [5].

Several points arising from the peculiarities of the representational space of quantum theory are important enough to mention. First of all, there is no consistent interpretation of each atom in the superposed state as two spatially separated physical entities moving over different trajectories in the gravitational field. Standard quantum theory prohibits such a view since the quantum state has no consistent spatial ontology. While the differential

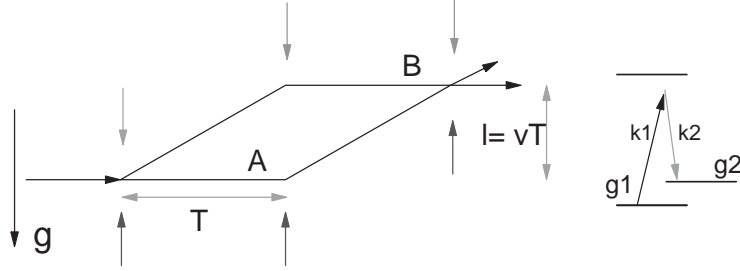


Figure 1: The space-time diagram of an atom interferometer. The laser pulses from a pair of Raman lasers, of duration  $\pi/2$ ,  $\pi$  and  $\pi/2$  in terms of Rabi frequency, create the superposition of hyperfine states that separate and recombine in space due to the differential momentum imparted. The differential momentum of the states  $p = \hbar(k_1 - k_2)$ .

phase at an end point can be measured, a direct measurement of the spatial position of each atom will result in one or the other spatial position. For the slowly moving atoms, it is possible to associate a wave that obeys the wave equation for evolution, but that is uniquely the de Broglie wave and not the Compton wave. This fact can be directly checked by subjecting the same atoms to a conventional double slit interference experiment and observing the ‘fringes’ which show the pattern corresponding to the ratio of the de Broglie wavelength  $\lambda_{dB} = 2\pi\hbar/m_i v$  to the slit separation. Hence, there is no empirical support to associate a Compton wave to the slow atoms in atom interferometry, *except as a conversion of the unit of mass*.

The geometry relevant for atom interferometry involving ground states  $|g1\rangle$  and  $|g2\rangle$  with energy difference  $\Delta E/\hbar = \nu_0$  and an upper state  $|e\rangle$  is indicated in figure 1 [8]. A two-laser system is used for Raman pulses, each detuned sufficiently from the excited state and adjusted to have a difference in their frequency equal to  $\nu_1 - \nu_2 = \nu_0$ . The laser pulses of duration  $\pi/2$ , relative to the inverse of the Rabi frequency  $\Omega$ , create the superposition of the ground states (hyperfine states). However,  $|g2\rangle$  is associated with an excess momentum  $\hbar k_1 - \hbar k_2 = \hbar \kappa$  and the entangled state ‘separate’ by  $l = \hbar \kappa T/m$  over duration  $T$ . A  $\pi$  laser pulse inverts the states and the momenta and the paths recombine after duration  $T$  at which point a  $\pi/2$  pulse creates similar states at the interferometer output. The gravitational phase difference is

simply the difference in phase accumulated over the path sections A and B with spatial separation  $l$  and temporal extent  $T$ , with gravitational potentials  $\phi_g(A)$  and  $\phi_g(B)$ .

$$\begin{aligned}\delta\Delta_g &= -m_g (\phi_g(x_1) - \phi_g(x_2)) T/\hbar = -m_g g l T/\hbar \\ &= -m_g g (\hbar \kappa T / m_i) T/\hbar = -\left(\frac{m_g}{m_i}\right) g \kappa T^2\end{aligned}\quad (9)$$

This is the expression for gravitational differential phase shift in gravimetry. All other phases, dynamical and laser induced, in the problem are equal for the two paths and drop out.

If the equivalence principle is assumed, then the ratio  $m_g/m_i$  drops out and we get the familiar expression,  $\delta\Delta_g = -\kappa g T^2$ . However, the general impression that the mass drops out from this expression is not correct since the mass term is just hidden in its relation to  $\kappa$ . Since  $\kappa = lm/\hbar T$  and  $l = vT$  where  $v$  is the relative velocity between the two wave packets,  $\kappa = mv/\hbar = 2\pi/\lambda_{dB}$ . The two-photon momentum difference replaces its equivalent relative momentum of the wave-packets, which is mass dependent. Here  $\lambda_{dB}$  is a reduced de Broglie wavelength,  $1/\lambda_{dB} = 1/\lambda_{dB1} - 1/\lambda_{dB2}$ . The differential phase shift can be written as

$$\delta\Delta_g = -\kappa g T^2 = -\frac{2\pi g T^2}{\lambda_{dB}} \quad (10)$$

*The phase shift is essentially the ratio of free fall distance in the gravitational field  $g$  and the de Broglie wavelength, as dictated by the equivalence principle.* In a picture projected to real space, the fringes from the interference of the de Broglie waves will ‘fall’ through a distance  $2gT^2$  over time  $2T$  and the phase shift is the ratio of this fringe shift and the ‘centre of mass de Broglie wavelength’  $h/(mv)/2 = 2\lambda_{dB}$ . It is clear that the gravitational phase is scaled to the de Broglie waves, as expected for slow non-relativistic atoms, and not to the relativistic and notional Compton wave.

Every expression in quantum theory that involves the nonrelativistic de Broglie wavelength  $\lambda = h/mv$  can be re-written in terms of the Compton wavelength with a change of unit from mass to frequency, as

$$\lambda = h/mv = \frac{hc^2}{mc^2v} = \frac{c^2}{\omega_c v} = \frac{c^2 T}{\omega_c l} \quad (11)$$

This is the Compton wavelength multiplied by the large factor  $cT/l$ , the ratio of the light travel distance to the spatial separation between the atomic states, reiterating the fact that it is not the Compton wave that is relevant in the experiment and the measurement. Then the relative phase shift, assuming the equality of the inertial and gravitational masses, is

$$\delta\Delta_g = -\omega_c g l T / c^2 \quad (12)$$

The appearance of the ‘relativistic’ Compton frequency is therefore fictitious. The explicit dependence of phase is on the de Broglie wavelength and not on the Compton wavelength, as revealed in equation 10. This can of course be empirically proved simply by forming a spatial interference pattern with the same slow atoms through a double slit, revealing the underlying relevant wavelength in the spatial interference pattern.

We stress the important point that a physical clock should admit standard clock operations relative to a primary standards and for this it is necessary that the oscillator phase is directly accessible for comparison. For the atom interferometer gravimeter, this phase is manifested in the population of either of the hyperfine states after recombination, determined by the *phase difference* imprinted gravitationally due to the difference in the interaction energy,  $-m(\phi_g(x_1) - \phi_g(x_2))$ . Hence we cannot treat each of the individual wavepackets as individual clocks, just as the two-state quantum superposition separated in a Stern-Gerlach magnet is not two individual physical systems. They do not even exist in space as physical reality, except in certain non-standard interpretations of quantum mechanics.

What constitutes a genuine clock and what is its ultimate precision in a measurement of gravitational time dilation is a hard problem in the context of the apparent universal behaviour of waves, quantum states and physical clocks in a gravitational field. Our analysis helps to provide a satisfactory answer that is rigorous in its treatment of the equivalence principle and the effect of weak gravitational fields and goes a long way in clarifying several issues. We have established that the gravitational phase shift in atom interferometry involves the non-relativistic momentum and the de Broglie wave rather than the Compton wave. The relation to the equivalence principle is brought out clearly by demonstrating how the ratio of the gravitational mass to inertial mass remains in the equations for the differential quantum phase. Moreover, we have argued that the observable oscillator is not the Compton wave because the relevant spatial fringe pattern is determined by

the de Broglie wavelength. The Compton wave has no physical manifestation in gravimetry with massive particles and the Compton frequency in the expression for phase shift is no more than a conversion of the unit of mass.

## References

- [1] A. Einstein, *Annalen der Physik* **35** 898 (1911) .
- [2] C. S. Unnikrishnan and G. T. Gillies, *Universality in the Gravitational Stretching of Clocks, Waves and Quantum States*, GRF honorable mention essay 2011, arXiv:1106.4151.
- [3] C. S. Unnikrishnan, *Mod. Phys Lett. A***17**, 1081 (2002) .
- [4] C. S. Unnikrishnan and G. T. Gillies, *Frontiers of Physics* **3**, 444 (2008).
- [5] M. de Angelis *et al.*, *Meas. Sci. Technol.* **20**, 022001 (2009).
- [6] H. Müller, A. Peters and S. Chu, *Nature* **463**, 926 (2011).
- [7] P. Wolf *et al.*, *Nature* **467**, E1 (2011).
- [8] M. Kasevich and S. Chu, *Appl. Phys. B* **54**, 321 (1992).
- [9] A. Peters, K. Chung and S. Chu, *Nature* **400**, 849 (1999).
- [10] H. Müller *et al.*, *Nature* **467**, E2 (2011) .
- [11] M. A. Hohensee et al, *J. Phys. Conf. series*, **264**, 012009 (2011).
- [12] S. Sinha and J. Samuel, *Atom interferometry and the gravitational red-shift*, arXiv:1102.2587 (2011).